

# Improving Two-Way Selective Decode-and-forward Wireless Relaying with Energy-Efficient One-bit Soft Forwarding

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## Abstract

Inspired by the application scenarios such as battery-operated wireless sensor networks (WSN), we investigate the design of an easy-to-implement energy-efficient two-way relaying scheme in which two source nodes exchange their messages with the help of an intermediate relay node. Due to their efficient decoder hardware implementation, block codes, such as Hamming and BCH codes, instead of convolutional and other sophisticated codes, are adopted in WSN-related standards, such as Bluetooth and IEEE802.15.6. At the network level, cooperative relaying with network coding can help reduce overall energy consumption. Motivated by these considerations, we address the challenge of improving the standard two-way selective decode-and-forward protocol (TW-SDF) with minor additional complexity and energy consumption. By following the principle of soft relaying, our solution is the two-way one-bit soft forwarding (TW-1bSF) protocol in which the relay forwards the one-bit quantization of a posterior information about the transmitted bits, associated with an appropriately designed reliability parameter. As the second main contribution, we derive tight upper bounds on the BLER performance for both the TW-SDF protocol and the TW-1bSF protocol, provided that the two-way relaying network applies block codes and hard decoding. The error probability analysis confirms the superiority of the TW-1bSF protocol over the TW-SDF protocol. In addition, our further derived asymptotic performance gain by TW-1bSF over TW-SDF suggests that the proposed protocol offers higher gain when longer block codes are used.

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## I. INTRODUCTION

Traditionally, the design of network protocols mainly focuses on how to maximize the throughput or bandwidth efficiency. Recently, the design of energy efficient wireless networks has received an explosion of interest. Instead of transmission power consumption, the aforementioned energy efficiency refers to the energy consumed by communication units to process signals, such as encoding and decoding. For example, in some remote monitoring applications using wireless sensor networks (WSN), the sensor nodes are expected to be low cost and have many years of battery life. It is clear that energy efficiency is the most critical design issue for such networks. Inspired by such application scenarios, in this paper we investigate the design of an easy-to-implement energy-efficient *two-way* relaying scheme in which two source nodes exchange their messages with the help of an intermediate relay node.

Applying the idea of network coding [1] to this network is particularly simple and fruitful. This is because network coding over the binary field (i.e. XOR) is sufficient for the topology, and exchanging one pair of messages between the two nodes requires only three packet transmissions, instead of four when using routing, under the setting of time division multiple access. Here, the three-transmission relaying method is referred to as *direct network coding* (DNC). Allowing the source nodes broadcast simultaneously, as proposed by physical-layer network coding (PNC) [2], [3] and analog network coding (ANC) [4], will further reduce the number of transmissions to two, resulting in even higher spectral efficiency. However, these two spectral-efficient schemes are not amenable for the wireless networks with the energy efficiency requirement because of the constraints like accurate synchronization, sophisticated decoding process at relay, no exploration of the direct link, and stringent channel estimation [5], [6].

Based on DNC, there are two variations of the two-way decode-and-forward protocols, namely, non-selective and selective. In the two-way (non-selective) decode-and-forward (DF) protocol, the relay decodes both packets received from the two sources and broadcasts the network-coded (XOR) version of the two decoded packets, regardless of whether two decoded packets are erroneous or not. Due to the effect of error propagation, the DF protocol incurs serious performance degradation. An modification to the DF protocol is the two-way selective decode-and-forward (TW-SDF) protocol. In this protocol, a sufficiently strong CRC code is used to detect the correctness of the two decoded packets at the relay. Then the relay only broadcasts the network-coded packet generated from two correct decoded packets. This approach effectively avoids error propagation, and can successfully recover the loss on the diversity order of the DF protocol.

In *one-way* relaying networks, the SDF protocol can be further improved by using the soft relaying

technique [7], [8], [9], [10], [11]. Specifically, the relay uses some soft decoder to derive the a posteriori probabilities of code bits, and then forwards these soft information values. At the desired receiver, these forwarded soft information values are exploited as a priori information by source-controlled soft decoders [12] to improve decoding performance. It is shown that performance improvement over the SDF can be achieved because in many occasions, an incorrectly decoded packet contains only few erroneous bits and the a posteriori information forwarded by the relay may help locate the position of these error bits. However, the main disadvantage of the soft relaying protocol is that it demands significantly additional bandwidth and/or power consumption, mainly due to the requirement of transmitting soft values or their multi-bit quantized version and the use of complicated soft-input soft-output decoders. The key motivation behind our work is to simplify the soft relaying method by using only the one-bit quantized representation of the soft information, and hence avoiding the complexity associated with soft signal forwarding and soft decoding. To suppress the error propagation due to the low-rate quantization as in the non-selective DF protocol, the quantized bit message is forwarded along with a reliability parameter, which is utilized by the decoder to estimate the equivalent LLR of the message from the relay [13]. The so-constructed *one-bit soft forwarding* (1bSF) protocol is almost as easy-to-implement and energy-efficient as the SDF protocol [14] (see also [15]). Through computer simulation, we have preliminarily demonstrated the effectiveness of the protocol for the one-way relay network in [16].

In this paper, we propose two-way 1bSF protocol, and then dedicate on performance analysis and comparison of the proposed TW-1bSF protocol and the TW-SDF protocol. In this paper, our contributions include (1) proposing the TW-1bSF protocol to improve the TW-SDF protocol while retaining the same energy efficiency and complexity, (2) deriving tight performance bound for the block-code-based TW-SDF relaying protocol for the first time, (3) analyzing the performance of the TW-1bSF, (4) deriving the asymptotic performance gain of the TW-1bSF protocol over the TW-SDF protocol, and (5) analyzing the design of reliability parameter to improve the TW-1bSF protocol.

**Notations:**  $d(\mathbf{y}) = |\{\tau \in \{1, 2, \dots, n\} : y_\tau < 0\}|$  is the number of negative signals within a signal vector  $\mathbf{y}$ ,  $d(\mathbf{y}_1, \mathbf{y}_2) = |\{\tau : y_{1,\tau}y_{2,\tau} < 0\}|$  denotes the number of signals with different signs between two signal vectors  $\mathbf{y}_1$  and  $\mathbf{y}_2$ ;  $\widehat{P}$  represents *upper* bound,  $\widetilde{P}$  represents *lower* bound.

## II. SYSTEM MODEL AND RELATED WORKS

Consider a three-node two-way wireless relaying network. In this network, via the help of a relay  $R$ , two sources  $S1$  and  $S2$  exchange their information messages, which are encoded to suppress channel distortions. Fig.1 shows the discrete-time model of the DNC-based two-way relaying over this network.

During the first time slot,  $S1$  broadcasts  $\mathbf{x}_1$  with power  $E_1$  to  $R$  and  $S2$ . The received signal vectors at  $R$  and  $S2$  are  $\mathbf{y}_{1r}$  and  $\mathbf{y}_{12}$ , respectively. During the second time slot,  $S2$  broadcasts  $\mathbf{x}_2$  with power  $E_2$ . The received signal vectors at  $R$  and  $S1$  are  $\mathbf{y}_{2r}$  and  $\mathbf{y}_{21}$ , respectively. After receiving the signals  $\mathbf{y}_{1r}$  and  $\mathbf{y}_{2r}$ , the relay generates a packet of signals  $\mathbf{x}_r(\mathbf{y}_{1r}, \mathbf{y}_{2r})$  for forwarding, where  $\mathbf{x}_r(\cdot, \cdot)$  is the mapping determined by relaying schemes and  $\mathbf{x}_r$  has unit power. During the third time slot,  $R$  broadcasts  $\mathbf{x}_r$  with power  $E_r$ . The signal vectors received at  $S1$  and  $S2$  are  $\mathbf{y}_{r1}$  and  $\mathbf{y}_{r2}$ , respectively.

In this system model,  $\mathbf{x}_i = [x_{i,1}, x_{i,2}, \dots, x_{i,n}] \in \mathcal{X}^{c(n)}$ ,  $i \in \{1, 2\}$  and  $x_{i,j} \in \{+1, -1\}$ , is the symbol codeword generated from a bit codeword  $\mathbf{c}_i$  by BPSK mapping  $x_{i,\tau} = 1 - 2c_{i,\tau}$ ,  $\tau \in \{1, 2, \dots, n\}$ ;  $\mathbf{c}_1$  and  $\mathbf{c}_2$  are encoded from the same codebook;  $\mathbf{y}_{ij}$  is the received signal at the node  $j \in \{1, 2, r\}$  ( $r$  for  $R$ , 1 for  $S1$ , 2 for  $S2$ ) from the node  $i$ ;  $h$ ,  $h_1$  and  $h_2$  are the channel coefficients of the  $S1$ - $S2$  link (direct link), the  $S1$ - $R$  link and the  $S2$ - $R$  link, respectively, and all are reciprocal;  $\mathbf{n}_{ij}$  represents independent identically-distributed (i.i.d.) Gaussian noises with  $\mathcal{N}(0, \sigma_j^2)$  at the node  $j$  when receiving from the  $i$ th node. We assume that the channel coefficients are fixed during the two-way relaying communication. We also assume all noises have the same variance  $\sigma_r^2 = \sigma_1^2 = \sigma_2^2 = N_0/2$ , and the sources utilize the same power  $E_1 = E_2 = E$ . For simplicity, we further assume  $N_0 = 1$ .

For the DNC-based two-way relaying method, it is crucial to design the mapping function  $\mathbf{x}_r(\mathbf{y}_{1r}, \mathbf{y}_{2r})$  such that both messages can be decoded by their desired receivers, and more desirable to optimize it to achieve the best performance in terms of capacity or bit error rate. By using the concept of SDF on designing  $\mathbf{x}_r(\mathbf{y}_{1r}, \mathbf{y}_{2r})$ , the TW-SDF protocol is conceived and is about to be described in the following.

#### A. Two-way SDF Relaying Protocol

The relay  $R$  decodes  $\mathbf{y}_{1r}$  and  $\mathbf{y}_{2r}$  at the first and second time slot, respectively, to generate  $\hat{\mathbf{x}}_{1r}$  and  $\hat{\mathbf{x}}_{2r}$ , the estimations of  $\mathbf{x}_1$  and  $\mathbf{x}_2$ . Then the relay combines the codewords  $\hat{\mathbf{x}}_{1r}$  and  $\hat{\mathbf{x}}_{2r}$  with network coding, and forwards the network-coded word  $\mathbf{x}_r$  to the sources during the third time slot. If any of  $\hat{\mathbf{x}}_{1r}$  and  $\hat{\mathbf{x}}_{2r}$  is erroneously decoded, the decoding error will be propagated from  $R$  to the sources, resulting in performance degradation. To eliminate such detrimental effect, TW-SDF keeps  $R$  from forwarding erroneous information, by designing the forwarded signal as

$$\mathbf{x}_r^{\text{TW-SDF}} = \begin{cases} \hat{\mathbf{x}}_{1r} \circ \hat{\mathbf{x}}_{2r}, & \text{if } \hat{\mathbf{x}}_{1r} = \mathbf{x}_1, \hat{\mathbf{x}}_{2r} = \mathbf{x}_2 \\ 0, & \text{o.w.} \end{cases} \quad (1)$$

where  $\circ$  denotes the Hadamard product, which is equivalent to the bit-wise XOR processing.

To demonstrate the decoding algorithm at the sources, without loss of generality, we focus on the receiving process at  $S2$ . After the third time slot,  $S2$  has the received signals  $\mathbf{y}_{12}$  and  $\mathbf{y}_{r2}$ . Among

them,  $\mathbf{y}_{12}$  is distorted by only Gaussian noise, while  $\mathbf{y}_{r2}$  is distorted by Gaussian noise and the transmitted codeword  $\mathbf{x}_2$  from  $S2$ . Since  $S2$  knows the transmitted codeword of its own, it can remove the contamination of  $\mathbf{x}_2$  from  $\mathbf{y}_{r2}$ , and generate a new detection statistic

$$\tilde{\mathbf{y}}_{r2} \triangleq \mathbf{x}_2 \circ \mathbf{y}_{r2} = h_2 \sqrt{E_r} \mathbf{x}_2 \circ \mathbf{x}_r^{\text{TW-SDF}} + \mathbf{x}_2 \circ \mathbf{n}_{r2} = h_2 \sqrt{E_r} \mathbf{x}_2 \circ \hat{\mathbf{x}}_{1r} \circ \hat{\mathbf{x}}_{2r} + \tilde{\mathbf{n}}_{r2} \quad (2)$$

where  $\tilde{\mathbf{n}}_{r2} = \mathbf{n}_{r2} \circ \mathbf{x}_2$  is a block of i.i.d. Gaussian noises with  $\mathcal{N}(0, N_0/2)$ . By summing  $\mathbf{y}_{12}$  and  $\tilde{\mathbf{y}}_{r2}$  with maximal ratio combining (MRC),  $S2$  can decode the codeword  $\mathbf{x}_1$  by

$$\hat{\mathbf{x}}_1^{\text{TW-SDF}} = \begin{cases} \mathfrak{D}(4h\sqrt{E}\mathbf{y}_{12} + 4h_2\sqrt{E_r}\tilde{\mathbf{y}}_{r2}), & \text{if } \hat{\mathbf{x}}_{1r} = \mathbf{x}_1, \hat{\mathbf{x}}_{2r} = \mathbf{x}_2 \\ \mathfrak{D}(4h\sqrt{E}\mathbf{y}_{12}), & \text{o.w.} \end{cases} \quad (3)$$

where  $\mathfrak{D}$  represents a decoding algorithm.

### III. TWO-WAY ONE-BIT SOFT FORWARDING (TW-1BSF) PROTOCOL

The objective of our work is to improve the TW-SDF protocol by using the soft relaying principle, while requiring no additional bandwidth and keeping the decoding algorithm simple such that the energy efficient expectation can be met. Provided that the decoding error occurs at  $R$ , our solution is to one-bit quantize the soft outputs of the relay decoder, which could be the soft output for message bit or code bit, and then forward the network-coded combination of the quantized bits to the sources. Since the a posterior soft output for each message/code bit, like LLR, is composed of a sign component and an reliability component, forwarding the one-bit quantization of the soft outputs is equal to conveying their sign information. To simplify the transmission of their reliability components,  $R$  generates a single reliability value for the forwarded packet, and then pass it to the receivers (similar to the equivalent SNR approach in [13]). By this approach, the transmission of the multiple reliability values is simplified to the transmission of a single soft value. The desired receiver uses the block of quantized bits and the associated reliability value as a priori information to enhance the decoding of the signal received through the direct link. This relaying approach is named as *two-way one-bit soft forwarding* (TW-1bSF) protocol.

We can implement the TW-1bSF protocol in a lot of ways. In this paper, we consider a simple implementation method in which  $\mathbf{x}_r^{\text{TW-1bSF}} = \hat{\mathbf{x}}_{1r} \circ \hat{\mathbf{x}}_{2r}$  is forwarded along with a reliability value, regardless of the correctness of  $\hat{\mathbf{x}}_{1r}$  and  $\hat{\mathbf{x}}_{2r}$ . Let  $\ell_{r2}$  be the LLR associated with  $\tilde{\mathbf{y}}_{r2}$  in TW-1bSF. Then the design of  $\ell_{r2}$  should be deliberated for three different cases, depending on whether decoding error occurs for  $\hat{\mathbf{x}}_{1r}$  and  $\hat{\mathbf{x}}_{2r}$  at  $R$ , and whether the forwarded packet  $\mathbf{x}_r$  is correctly decoded at the receiver. Given  $\mathfrak{D}(\mathbf{y})$  being a hard-output decoder and  $\mathfrak{D}(\mathbf{y}) \in \mathcal{X}^{c(n)}$  for  $\forall \mathbf{y}$ , we refer to erroneous decoding

events as

$$\begin{aligned}\mathcal{E} &= \{\hat{x}_1 \neq x_1\}, \quad \mathcal{E}_{12} = \{\mathcal{D}(\mathbf{y}_{12}) \neq x_1\}, \quad \mathcal{E}_{1r} = \{\hat{x}_{1r} \neq x_1\} \\ \mathcal{E}_{2r} &= \{\hat{x}_{2r} \neq x_2\}, \quad \mathcal{E}_{r2} = \{\mathcal{D}(\tilde{\mathbf{y}}_{r2}) \neq x_r \circ x_2\}.\end{aligned}\quad (4)$$

In addition, let  $A^c$  denote the complement of an error event  $A$ , i.e. a correct event, for example,  $\mathcal{E}^c = \{\hat{x}_1 = x_1\}$ . In the following, we introduce the design of  $\ell_{r2}$  for three different cases.

- **Case I:** It is the case when  $\hat{x}_{1r}$  and  $\hat{x}_{2r}$  are both correctly decoded, i.e.,  $\hat{x}_{1r} = x_1$  and  $\hat{x}_{2r} = x_2$ , and is denoted by  $\Theta_I \triangleq \mathcal{E}_{12} \cap \mathcal{E}_{1r}^c \cap \mathcal{E}_{2r}^c \cap \mathcal{E}_{r2}$ . In this case,  $\ell_{r2}$  at  $S2$  is

$$\ell_{r2} = 4h_2\sqrt{E_r}\tilde{\mathbf{y}}_{r2}; \quad (5)$$

- **Case II:** It is the case when  $\hat{x}_{1r}$  and  $\hat{x}_{2r}$  are not both correctly decoded, i.e.,  $\mathcal{E}_{1r} \cup \mathcal{E}_{2r}$ , but the forwarded signal  $x_r$  is correctly decoded at  $S2$ , i.e.,  $x_r = \mathcal{D}(\mathbf{y}_{r2})$ , and is denoted by  $\Theta_{II} \triangleq \mathcal{E}_{12} \cap (\mathcal{E}_{1r} \cup \mathcal{E}_{2r}) \cap \mathcal{E}_{r2}^c$ . This case can be further decomposed into three disjoint subcases as  $\Theta_{II} = \Theta_{IIa} \cup \Theta_{IIb} \cup \Theta_{IIc}$ , where  $\Theta_{IIa} \triangleq \mathcal{E}_{12} \cap \mathcal{E}_{1r} \cap \mathcal{E}_{2r}^c \cap \mathcal{E}_{r2}^c$ ,  $\Theta_{IIb} \triangleq \mathcal{E}_{12} \cap \mathcal{E}_{1r}^c \cap \mathcal{E}_{2r} \cap \mathcal{E}_{r2}^c$  and  $\Theta_{IIc} \triangleq \mathcal{E}_{12} \cap \mathcal{E}_{1r} \cap \mathcal{E}_{2r} \cap \mathcal{E}_{r2}^c$ . In this case, we let

$$\ell_{r2} = \mathcal{D}(\tilde{\mathbf{y}}_{r2})\mathcal{L} = (x_r \circ x_2)\mathcal{L} \quad (6)$$

where  $\mathcal{L}$  is the reliability value for the forwarded packet  $x_r$ ;

- **Case III:** It is the case when  $\hat{x}_{1r}$  and  $\hat{x}_{2r}$  are not both correctly decoded, and  $x_r$  is not correctly decoded at  $S2$ , and is denoted by  $\Theta_{III} \triangleq \mathcal{E}_{12} \cap (\mathcal{E}_{1r} \cup \mathcal{E}_{2r}) \cap \mathcal{E}_{r2}$ . In this case, we let

$$\ell_{r2} = \mathcal{D}(\tilde{\mathbf{y}}_{r2})\mathcal{L}^- \quad (7)$$

where  $\mathcal{L}^-$  is the reliability value of  $\mathcal{D}(\tilde{\mathbf{y}}_{r2})$  for Case III.

As for  $\mathcal{L}$  and  $\mathcal{L}^-$ , by principle, they should be inversely proportional to the codeword error rate. By using the fact that the codeword error rate is a monotonic increasing function of the channel bit error rate, and then the relationship between bit error rate and LLR as well as the LLR operation rule [12], we can assign the reliability value for Case II as

$$\mathcal{L} = \min \left( \log \frac{1-p_{1r}}{p_{1r}}, \log \frac{1-p_{2r}}{p_{2r}} \right) \quad (8)$$

where  $p_{1r} = Q(\sqrt{2h_1^2 E})$  is the channel bit error rate of the  $S1$ - $R$  link, and  $p_{2r} = Q(\sqrt{2h_2^2 E})$  is the channel bit error rate of the  $S2$ - $R$  link, in which  $Q(\cdot)$  is the Gaussian Q function. Similarly, the reliability for Case III can be designed as

$$\mathcal{L}^- = \log \frac{1 - (1 - (1 - p_{1r})(1 - p_{2r}))}{1 - (1 - p_{1r})(1 - p_{2r})} = \log \frac{(1 - p_{1r})(1 - p_{2r})}{1 - (1 - p_{1r})(1 - p_{2r})} \quad (9)$$

where  $1 - (1 - p_{1r})(1 - p_{2r})$  is the probability that at least one of channel bits from the  $S1-R$  and the  $S2-R$  links is in error.

By approximating  $\ell_{r2}$  as the LLR of the  $S1-R-S2$  path, at  $S2$  the received signal vector  $\mathbf{y}_{12}$  can be decoded by using the decoding method of [12] into  $\hat{\mathbf{x}}_1 = \arg \max_{\mathbf{x} \in \mathcal{X}^{c(n)}} \left( \langle 4h\sqrt{E}\mathbf{y}_{12}, \mathbf{x} \rangle + \langle \ell_{r2}, \mathbf{x} \rangle \right) = \arg \min_{\mathbf{x} \in \mathcal{X}^{c(n)}} \|4h\sqrt{E}\mathbf{y}_{12} + \ell_{r2} - \mathbf{x}\|$  where

$$\ell_{r2} = \begin{cases} 4h_2\sqrt{E_r}\tilde{\mathbf{y}}_{r2}, & \text{Case I} \\ (\mathbf{x}_r \circ \mathbf{x}_2)\mathcal{L}, & \text{Case II} \\ \mathfrak{D}(\tilde{\mathbf{y}}_{r2})\mathcal{L}^-, & \text{Case III} \end{cases} \quad (10)$$

#### A. Selection of Channel Code and Decoder for TW-1bSF

Many channel codes can be applied in the proposed TW-1bSF protocol. However, for the WSN consisting of devices with limited energy and computation capability, simple block codes like Hamming code or BCH code are more preferable, in particular, the BCH codes with the syndrome decoder using the Berlekamp-Massey (BM) and Chien's search (CS) algorithm is 15% more energy efficient than the best performing convolutional codes [17]. Therefore, in this paper, we consider the TW-1bSF protocol applying Hamming code or BCH code, and hard decoder at the receiver ends. Then the decoding at  $S2$  can be rewritten as

$$\hat{\mathbf{x}}_1^{\text{TW-1bSF}} = \mathfrak{D} \left( 4h\sqrt{E}\mathbf{y}_{12} + \ell_{r2} \right) = \mathfrak{D}(\ell_d) \quad (11)$$

where  $\ell_d \triangleq 4h\sqrt{E}\mathbf{y}_{12} + \ell_{r2}$ .

For most of practical block codes, the minimum distance is an odd number, conventionally denoted by  $2t + 1$ , where  $t$  is the correcting capacity. Therefore, in this paper we consider the block codes with minimum distance  $2t + 1$ .

### IV. PERFORMANCE ANALYSIS OF TW-SDF AND TW-1BSF PROTOCOLS

To achieve energy-efficient relaying, as aforementioned, we consider block codes and hard decoding. In this section, we study the performance of the TW-SDF and the TW-1bSF protocols in terms of block error rate (BLER).

*Definition 1.* Without loss of generality, our focus is on the receiving at  $S2$ , so the BLER analyzed in this paper is the error probability of codewords transmitted from  $S1$  to  $S2$ , defined as

$$P^{(\mathbf{x}_1)} \triangleq \Pr(\hat{\mathbf{x}}_1 \neq \mathbf{x}_1). \quad (12)$$



### A. Background

In the considered two-way relaying network, there are four point-to-point links, i.e.,  $S1 \rightarrow S2$ ,  $S1 \rightarrow R$ ,  $S2 \rightarrow R$  and  $R \rightarrow S2$ , which are denoted as  $\{12, 1r, 2r, r2\}$  respectively. They have channel bit error rates  $p_{12} = Q(\sqrt{2h^2E})$ ,  $p_{1r} = Q(\sqrt{2h_1^2E})$ ,  $p_{2r} = Q(\sqrt{2h_2^2E})$ , and  $p_{r2} = Q(\sqrt{2h_2^2E_r})$ . We first identify the BLER of these links. Regardless of specific hard decoding method, we can upper and lower bound the BLER of a link by using the performance of the *bounded distance decoder* (BDD). Counting decoding error only if the received word falls *outside* of the decoding sphere of the *transmitted* codeword [18], we have the upper bound

$$\widehat{P}_i = 1 - \sum_{k=0}^t \binom{n}{k} p_i^k (1 - p_i)^{(n-k)}. \quad (13)$$

where  $i \in \{12, 1r, 2r, r2\}$ . By contrast, counting error only if the received words falling *inside* the decoding spheres of the *wrong* codewords, we have the lower bound

$$\widetilde{P}'_i = \sum_{k=2t+1}^n \beta'_k(p_i) = \sum_{k=2t+1}^n A_k \sum_{m=0}^t \sum_{j=0}^{\min(m, n-k)} \binom{k}{m-j} \binom{n-k}{j} p_i^{k-m+2j} (1 - p_i)^{n-k+m-2j} \quad (14)$$

where  $A_k$  is the number of the codewords with Hamming weight  $k$ ;  $\beta'_k(p_i)$  is the probability that the received word is decoded as a codeword whose Hamming distance from the transmitted codeword is  $k$ .

Except for perfect codes, the lower bound aforementioned is too loose to predict the BLER of any practical hard decoder. In this paper, we propose another lower bound which is decoder-oriented and tighter. Suppose the all-zero codeword is transmitted, since the correcting capability is  $t$ , the received words of Hamming weight  $t + 1$  are not ensured to be correctly decoded, and which wrong codewords they will be decoded to is determined by decoding method. Denote  $W_k$  as the number of the received words of weight  $t + 1$  which are decoded to wrong codewords of weight  $k$ . We name  $W_k$ ,  $2t + 1 \leq k \leq n$  as *sphere partitioning function* (SPF), which partitions the radius- $(t + 1)$  sphere centered at the transmitted zero codeword. Assume  $W_k$  is known for the hard decoder  $\mathfrak{D}(\cdot)$ , we have a tighter lower bound on the BLER of the decoder by the following lemma.

**Lemma 1** (Intrinsic Lower Bound of Hard Decoder). Give a hard decoder of SPF  $W_k$  and a link of channel BER  $p_i$ , the BLER of the link using the decoder is lower bounded by

$$\begin{aligned} \widetilde{P}_i &= \beta'_{2t+1}(p_i) + \left( W_{2t+1} - A_{2t+1} \binom{2t+1}{t} \right) p_i^{t+1} (1 - p_i)^{n-t-1} + \sum_{k=2t+2}^n \beta'_k(p_i) + W_k p_i^{t+1} (1 - p_i)^{n-t-1} \\ &\triangleq \sum_{k=2t+1}^n \beta_k(p_i). \end{aligned} \quad (15)$$



*Proof:* Given the transmission of the zero codeword, the loss lower bound (14) doesn't examine all the received words of weight  $t + 1$ . Among them,  $A_{2t+1} \binom{2t+1}{t}$  are decoded as wrong codewords of weight  $2t + 1$  while none is decoded to wrong codewords of weight  $k > 2t + 1$ . In contrast, given a hard decoder and its SPF  $W_k$ , among the received words of weight  $t + 1$  without examined by (14),  $W_{2t+1} - A_{2t+1} \binom{2t+1}{t}$  words are decoded to wrong codewords of weight  $2t + 1$  and  $W_k$  words are decoded to wrong codewords of weight  $k > 2t + 1$ . So lemma is proved. ■

### B. Preliminary results

Suppose, in the two-way rely network, all receivers utilize the same hard decoder with SPF  $W_k$ . Due to the equiprobability of codewords and the linearity of block codes, the BLER of  $\mathbf{x}_1$  at  $S_2$  equals to the conditional BLER given the transmission of  $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{1}$ , which corresponds zero codewords  $\mathbf{c}_1 = \mathbf{c}_2 = \mathbf{0}$ . For simplicity, in the rest of this paper the condition of  $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{1}$  is omitted. By decomposing the TW-SDF and the TW-1bSF protocols into disjoint cases, the BLERs of them can be written, respectively, as

$$P_{\text{TW-SDF}}^{(\mathbf{x}_1)} = P(\mathcal{E}, \Theta_{\text{I}}) + P(\Theta_{\text{II}} \cup \Theta_{\text{III}}) \quad (16)$$

and

$$P_{\text{TW-1bSF}}^{(\mathbf{x}_1)} = P(\mathcal{E}, \Theta_{\text{I}} \cup \Theta_{\text{II}} \cup \Theta_{\text{III}}) = P(\mathcal{E}, \Theta_{\text{I}}) + P(\mathcal{E}, \Theta_{\text{II}}) + P(\mathcal{E}, \Theta_{\text{III}}). \quad (17)$$

Upon the condition of  $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{1}$ , there are three properties which will facilitate the following BLER analysis. They are:

- 1) A channel bit is *in error* if the corresponding signal  $y_\tau < 0$  for  $\tau \in \{1, \dots, n\}$ , i.e.,  $d(y_\tau) = 1$ ;
- 2) The received signal  $\mathbf{y}$  is decoded erroneously means that the decoded codeword has weight no less than  $2t + 1$ , i.e.,  $d(\mathcal{D}(\mathbf{y})) \geq 2t + 1$ ;
- 3) When the BDD is applied to the link  $i$ ,  $\mathbf{y}_i$  would be correctly decoded if  $d(\mathbf{y}_i) \leq t$ , otherwise decoded in error.

By contrast to the notions in (4), if the link  $i$  uses BDD, the decoding error event is denoted as  $\dot{\mathcal{E}}_i$ , and then the probability of this error event is  $P(\dot{\mathcal{E}}_i) = \widehat{P}_i$  as given by (13). We also denote

$$\dot{\Theta}_j \triangleq \{\Theta_j : \mathcal{D} \text{ is BDD for all links}\} \quad (18)$$

with  $j$  representing cases, for instance  $\dot{\Theta}_{\text{I}} = \dot{\mathcal{E}}_{12} \cap \dot{\mathcal{E}}_{1r}^c \cap \dot{\mathcal{E}}_{2r}^c \cap \dot{\mathcal{E}}_{r2}$ .

We start with the analysis of  $P(\mathcal{E}, \Theta_{\text{I}})$  since it is the common part of  $P_{\text{TW-SDF}}^{(\mathbf{x}_1)}$  and  $P_{\text{TW-1bSF}}^{(\mathbf{x}_1)}$ . The event of  $\Theta_{\text{I}}$  is decoder-dependent, and its performance is usually hard to tract, if not impossible. Alternatively,

we examine its decoder-independent counterpart,  $\dot{\Theta}_I$ , and derive an upper bound of  $P(\mathcal{E}, \dot{\Theta}_I)$ . According to the definition of  $\dot{\Theta}_I$ , we have  $P(\mathcal{E}, \dot{\Theta}_I) = P(\dot{\Theta}_I) - P(\mathcal{E}^c, \dot{\Theta}_I) = \bar{P}_{12} (1 - \bar{P}_{1r})(1 - \bar{P}_{2r}) \bar{P}_{r2} - P(\mathcal{E}^c, \dot{\Theta}_I)$ , among which  $P(\mathcal{E}^c, \dot{\Theta}_I)$  is lower bounded by the following theorem.

*Theorem 1.* Given any hard decoder applied for decoding  $\ell_d$ ,  $P(\mathcal{E}^c, \dot{\Theta}_I)$  is lower bounded by

$$\begin{aligned} \bar{P}(\mathcal{E}^c, \dot{\Theta}_I) &\triangleq (1 - \bar{P}_{1r})(1 - \bar{P}_{2r}) \sum_{k=t+1}^{n-1} \binom{n}{k} \sum_{m=\max(0, t+1-(n-k))}^t \sum_{i=t+1}^{m+n-k} \binom{k}{m} \binom{n-k}{i-m} p_I^{n-k-(i-m)} q_I^m \\ &\quad \times \sum_{g=0}^{t-m} \sum_{j=0}^g \binom{k-m}{j} \dot{p}_I^j \dot{q}_I^{k-m-j} \binom{i-m}{g-j} \ddot{p}_I^{g-j} \ddot{q}_I^{i-m-(g-j)} \end{aligned} \quad (19)$$

where

$$\begin{aligned} p_I &= (1 - p_{r2})(1 - p_{12}), \quad q_I = p_{12}p_{r2} \\ \dot{p}_I &= \int_0^\infty Q\left(h_2\sqrt{2E_r} + \frac{h\sqrt{2E}}{h_2\sqrt{E_r}}x\right) \frac{1}{\sqrt{\pi}} \exp\left(-(x - h\sqrt{E})^2\right) dx, \quad \dot{q}_I = p_{r2}(1 - p_{12}) - \dot{p}_I \\ \ddot{p}_I &= \int_0^\infty Q\left(h\sqrt{2E} + \frac{h_2\sqrt{2E_r}}{h\sqrt{E}}x\right) \frac{1}{\sqrt{\pi}} \exp\left(-(x - h_2\sqrt{E_r})^2\right) dx, \quad \ddot{q}_I = p_{12}(1 - p_{r2}) - \ddot{p}_I. \end{aligned} \quad (20)$$

Note that  $\dot{p}$  and  $\ddot{p}$  are bivariate normal cumulative distribution functions, which are convenient to be calculated numerically.

*Proof:* In  $\dot{\Theta}_I$ , because of  $\dot{\mathcal{E}}_{1r}^c$  and  $\dot{\mathcal{E}}_{2r}^c$ , it has  $\tilde{\mathbf{y}}_{r2} = \tilde{\mathbf{y}}_{r2,I} \triangleq h_2\sqrt{E_r}\mathbf{x}_1 + \tilde{\mathbf{n}}_{r2}$  according to (2), so  $\dot{\mathcal{E}}_{r2}$  means that  $\tilde{\mathbf{y}}_{r2,I}$  has  $d(\tilde{\mathbf{y}}_{r2,I}) = k \geq t+1$  bits in error. Moreover, it is clear  $\dot{\mathcal{E}}_{12}$  means that  $\mathbf{y}_{12}$  has  $i \geq t+1$  bits in error. Let  $\bar{P}(\mathcal{E}^c, \dot{\Theta}_I) \triangleq P(\dot{\mathcal{E}}^c, \dot{\Theta}_I) = P(d(\ell_d) \leq t, \dot{\Theta}_I)$ , we have the inequality

$$\begin{aligned} P(\mathcal{E}^c, \dot{\Theta}_I) &\geq \bar{P}(\mathcal{E}^c, \dot{\Theta}_I) \\ &= (1 - \bar{P}_{1r})(1 - \bar{P}_{2r}) \sum_{k=t+1}^{n-1} \binom{n}{k} P(d(\ell_d) \leq t, d(\tilde{\mathbf{y}}_{r2,I}) = k, d(\mathbf{y}_{12}) \geq t+1). \end{aligned} \quad (21)$$

Note here  $k = n$  is omitted because of  $P(d(\ell_d) \leq t, d(\tilde{\mathbf{y}}_{r2,I}) = n, d(\mathbf{y}_{12}) \geq t+1) = 0$ . Suppose that, among the  $i$  erroneous bits of  $\mathbf{y}_{12}$ , there are  $m$  bits overlapping with the erroneous bits of  $\tilde{\mathbf{y}}_{r2}$ , as shown in Fig.2. It must be satisfied  $r \leq t$  such that  $d(\ell_d) \leq t$  is possible. Let  $(\tilde{\mathbf{y}}_{r2,I}, \mathbf{y}_{12})_{(k,m,i)}$  denote the events in which  $\tilde{\mathbf{y}}_{r2,I}$  and  $\mathbf{y}_{12}$  have  $k$  and  $i$  erroneous channel bits, respectively, and among them  $m$  erroneous bits are overlapped. Given the signal  $\tilde{\mathbf{y}}_{r2,I}$  of weight  $k$ , the number of such  $\mathbf{y}_{12}$  with the 2-tuple  $(m, i)$  is  $\binom{k}{m} \binom{n-k}{i-m}$ . So,

$$\begin{aligned} &P(d(\ell_d) \leq t, d(\tilde{\mathbf{y}}_{r2,I}) = k, d(\mathbf{y}_{12}) \geq t+1) \\ &= \sum_{m=\max(0, t+1-(n-k))}^t \sum_{i=t+1}^{m+n-k} \binom{k}{m} \binom{n-k}{i-m} P(d(\ell_d) \leq t, (\tilde{\mathbf{y}}_{r2,I}, \mathbf{y}_{12})_{(k,m,i)}). \end{aligned} \quad (22)$$

The derivation of  $P(d(\ell_d) \leq t, (\tilde{\mathbf{y}}_{r2,I}, \mathbf{y}_{12})_{(k,m,i)})$  is stemmed from the fact  $d(\ell_d) \leq t$  occurs if there are  $g \leq t - m$  erroneous channel bits among the bits  $\{\tau : \tilde{y}_{r2,I,\tau} < 0, y_{12,\tau} > 0\}$  and  $\{\tau : \tilde{y}_{r2,I,\tau} > 0, y_{12,\tau} < 0\}$ . Suppose that  $j$  bits of the  $g$  erroneous bits are within the set  $\{\tau : \tilde{y}_{r2,I,\tau} < 0, y_{12,\tau} > 0\}$ , and the rest erroneous bits are within the set  $\{\tau : \tilde{y}_{r2,I,\tau} > 0, y_{12,\tau} < 0\}$ , as shown in Fig. 2. For the  $\tau$ -th channel bit, let  $p_I = P(\tilde{y}_{r2,I,\tau} > 0, y_{12,\tau} > 0) = (1 - p_{r2})(1 - p_{12})$ , and  $q_I = P(\tilde{y}_{r2,I,\tau} < 0, y_{12,\tau} < 0) = p_{12}p_{r2}$ , and  $\dot{p}_I = P(\ell_{d,\tau} < 0, \tilde{y}_{r2,I,\tau} < 0, y_{12,\tau} > 0) = \Pr(h_2\sqrt{E_r}\tilde{y}_{r2,I,\tau} + h\sqrt{E}y_{12,\tau} < 0, y_{12,\tau} > 0)$ , and  $\dot{q}_I = P(\ell_{d,\tau} > 0, \tilde{y}_{r2,I,\tau} < 0, y_{12,\tau} > 0) = p_{r2}(1 - p_{12}) - \dot{p}_I$ , and  $\ddot{p}_I = P(\ell_{d,\tau} < 0, \tilde{y}_{r2,I,\tau} > 0, y_{12,\tau} < 0) = \Pr(h_2\sqrt{E_r}\tilde{y}_{r2,I,\tau} + h\sqrt{E}y_{12,\tau} < 0, \tilde{y}_{r2,I,\tau} > 0)$ , and  $\ddot{q}_I = P(\ell_{d,\tau} > 0, \tilde{y}_{r2,I,\tau} > 0, y_{12,\tau} < 0) = p_{12}(1 - p_{r2}) - \ddot{p}_I$ , where  $\ell_{d,\tau} = 4h\sqrt{E}y_{12,\tau} + 4h_2\sqrt{E_r}\tilde{y}_{r2,I,\tau}$  with  $y_{12,\tau} \sim \mathcal{N}(h\sqrt{E}, \frac{1}{2})$  and  $\tilde{y}_{r2,I,\tau} \sim \mathcal{N}(h_2\sqrt{E_r}, \frac{1}{2})$ . By counting all combinations of  $g$  and  $j$ , we get

$$P(d(\ell_d) \leq t, (\tilde{\mathbf{y}}_{r2,I}, \mathbf{y}_{12})_{(k,m,i)}) = p_I^{n-k-(i-m)} q_I^m \sum_{g=0}^{t-m} \sum_{j=0}^g \binom{k-m}{j} \dot{p}_I^j \dot{q}_I^{k-m-j} \ddot{p}_I^{g-j} \ddot{q}_I^{i-m-(g-j)}. \quad (23)$$

Finally, by substituting (23) into (22), and then into (21), we get (19), and hence prove the theorem. ■

By using the theorem, we upper bound  $P(\mathcal{E}, \Theta_I)$  by

$$P(\mathcal{E}, \Theta_I) \leq \hat{P}(\mathcal{E}, \Theta_I) \triangleq \hat{P}_{12} (1 - \hat{P}_{1r})(1 - \hat{P}_{2r}) \hat{P}_{r2} - \check{P}(\mathcal{E}^c, \Theta_I). \quad (24)$$

### C. Performance of TW-SDF protocol

The *one-way* SDF protocol has been well studied in terms of outage probability, symbol error rate, and ergodic capacity in the scenario without considering channel coding inside the relaying networks, but few results are available in the scenario with the application of channel coding inside the relaying networks [13]. In [13], a pair of BLER bounds are derived for the cooperative DF protocol [19] when applying convolutional codes and *soft* decoding. In this section, we derive an upper bound for the DNC-based TW-SDF protocol applying block codes and *hard* decoding.

*Theorem 2.* Given any hard decoder, the BLER of the TW-SDF protocol is upper bounded by

$$P_{\text{TW-SDF}}^{(\mathbf{x}_1)} \leq \hat{P}_{\text{TW-SDF}}^{(\mathbf{x}_1)} \triangleq P(\Theta_{\text{II}} \cup \Theta_{\text{III}}) + \hat{P}(\mathcal{E}, \Theta_I) \quad (25)$$

where  $\hat{P}(\mathcal{E}, \Theta_I) \triangleq \hat{P}_{12} (1 - \check{P}'_{1r})(1 - \check{P}'_{2r}) \hat{P}_{r2} - \check{P}(\mathcal{E}^c, \Theta_I)$ , which is different with  $\hat{P}(\mathcal{E}, \Theta_I)$  in (24).

*Proof:* First, we upper bound  $P(\Theta_{\text{II}} \cup \Theta_{\text{III}})$  by

$$P(\Theta_{\text{II}} \cup \Theta_{\text{III}}) = P(\mathcal{E}_{12})(1 - P(\mathcal{E}_{1r}^c)P(\mathcal{E}_{2r}^c)) \leq \hat{P}_{12} (1 - (1 - \hat{P}_{1r})(1 - \hat{P}_{2r})) = P(\Theta_{\text{II}} \cup \Theta_{\text{III}}). \quad (26)$$

Second, recall that  $\Theta_I = \mathcal{E}_{12} \cap \mathcal{E}_{1r}^c \cap \mathcal{E}_{2r}^c \cap \mathcal{E}_{r2}$ , because  $\mathcal{E}_{12} \subset \dot{\mathcal{E}}_{12}$  and  $\mathcal{E}_{r2} \subset \dot{\mathcal{E}}_{r2}$ , we have

$$\begin{aligned}
 P(\mathcal{E}, \Theta_I) &\leq P(\mathcal{E}, \dot{\mathcal{E}}_{12} \cap \mathcal{E}_{1r}^c \cap \mathcal{E}_{2r}^c \cap \dot{\mathcal{E}}_{r2}) = P(\dot{\mathcal{E}}_{12} \cap \mathcal{E}_{1r}^c \cap \mathcal{E}_{2r}^c \cap \dot{\mathcal{E}}_{r2}) - P(\mathcal{E}^c, \dot{\mathcal{E}}_{12} \cap \mathcal{E}_{1r}^c \cap \mathcal{E}_{2r}^c \cap \dot{\mathcal{E}}_{r2}) \\
 &\stackrel{(a)}{\leq} \widehat{P}_{12} (1 - \widetilde{P}'_{1r})(1 - \widetilde{P}'_{2r}) \widehat{P}_{r2} - P(\mathcal{E}^c, \dot{\Theta}_I) \\
 &\leq \widehat{P}_{12} (1 - \widetilde{P}'_{1r})(1 - \widetilde{P}'_{2r}) \widehat{P}_{r2} - \widetilde{P}(\mathcal{E}^c, \dot{\Theta}_I) \\
 &= \widehat{P}(\mathcal{E}, \Theta_I)
 \end{aligned} \tag{27}$$

At the step (a) above, we use the facts  $P(\mathcal{E}_{1r}^c) \leq (1 - \widetilde{P}'_{1r})$ ,  $P(\mathcal{E}_{2r}^c) \leq (1 - \widetilde{P}'_{2r})$ , as well as  $P(\mathcal{E}^c, \dot{\mathcal{E}}_{12} \cap \mathcal{E}_{1r}^c \cap \mathcal{E}_{2r}^c \cap \dot{\mathcal{E}}_{r2}) \geq P(\mathcal{E}^c, \dot{\Theta}_I)$  due to  $\dot{\Theta}_I \subset \dot{\mathcal{E}}_{12} \cap \mathcal{E}_{1r}^c \cap \mathcal{E}_{2r}^c \cap \dot{\mathcal{E}}_{r2}$ . Finally, combining the above two inequalities proves the theorem. ■

Note the extension of the following result to the one-way SDF protocol is straightforward.

#### D. Performance of TW-1bSF protocol

To derive the exact BLER performance the TW-1bSF protocol for a given hard decoder is generally formidable, except for some special hard decoders like BDD. Alternatively, we resort to derive upper bounds on the performance. Simply, we can universally upper bound  $P_{\text{TW-1bSF}}^{(x_1)} = P(\mathcal{E}, \Theta_I \cup \Theta_{II} \cup \Theta_{III})$  by  $\max_{\mathfrak{D}} P(\mathcal{E}, \Theta_I \cup \Theta_{II} \cup \Theta_{III})$  over  $\mathfrak{D} \in \{\text{hard decoders performing better than BDD}\}$ . It is clear that the maximization obtains when  $\mathfrak{D}$  is BDD, i.e., all links and  $\ell_d$  are decoded by BDD. However, this upper bound is very lossy for most practical decoders. In this section, given a specific hard decoder, we propose a tighter upper bound.

*Lemma 2* (Performance Upper Bound of TW-1bSF). Given any hard decoder, we have

$$P_{\text{TW-1bSF}}^{(x_1)} = P(\mathcal{E}, \Theta_I \cup \Theta_{II} \cup \Theta_{III}) \leq P(\mathcal{E}, \dot{\Theta}_I) + P(\mathcal{E}, \dot{\Theta}_{IIa}) + P(\mathcal{E}, \dot{\Theta}_{IIb}) + P(\dot{\Theta}_{IIc}) + P(\dot{\Theta}_{III}). \tag{28}$$

*Proof:* Given a hard decoder  $\mathfrak{D}$  which performs better than the BDD, it is clear that  $\dot{\mathcal{E}}_i^c \subset \mathcal{E}_i^c$  and  $\mathcal{E}_i \subset \dot{\mathcal{E}}_i$ , for all  $i$ , therefore,  $\Theta_I \cup \Theta_{II} \cup \Theta_{III} = (\mathcal{E}_{1r}^c \cap \mathcal{E}_{2r}^c \cap \mathcal{E}_{r2}^c)^c \cap \mathcal{E}_{12} \subset (\dot{\mathcal{E}}_{1r}^c \cap \dot{\mathcal{E}}_{2r}^c \cap \dot{\mathcal{E}}_{r2}^c)^c \cap \dot{\mathcal{E}}_{12} = \dot{\Theta}_I \cup \dot{\Theta}_{II} \cup \dot{\Theta}_{III}$ . From  $\Theta_I \cup \Theta_{II} \cup \Theta_{III} \subset \dot{\Theta}_I \cup \dot{\Theta}_{II} \cup \dot{\Theta}_{III}$ , we have  $P(\mathcal{E}, \Theta_I \cup \Theta_{II} \cup \Theta_{III}) \leq P(\mathcal{E}, \dot{\Theta}_I \cup \dot{\Theta}_{II} \cup \dot{\Theta}_{III}) \leq P(\mathcal{E}, \dot{\Theta}_I) + P(\mathcal{E}, \dot{\Theta}_{IIa}) + P(\mathcal{E}, \dot{\Theta}_{IIb}) + P(\dot{\Theta}_{IIc}) + P(\dot{\Theta}_{III})$ , which proves the lemma. ■

In the following, we analyze the terms of (28), and then the upper bound of  $P_{\text{TW-1bSF}}^{(x_1)}$ .

1)  $P(\mathcal{E}, \dot{\Theta}_{IIa})$ : To gain an upper bound of  $P(\mathcal{E}, \dot{\Theta}_{IIa}) = P(\dot{\Theta}_{IIa}) - P(\mathcal{E}^c, \dot{\Theta}_{IIa})$ , we derive a lower bound of  $P(\mathcal{E}^c, \dot{\Theta}_{IIa})$  by the theorem below.

*Theorem 3.* Given a hard decoder of SPF  $W_k$ ,  $P(\mathcal{E}^c, \dot{\Theta}_{\Pi a})$  is lower bounded by

$$\begin{aligned} \tilde{P}(\mathcal{E}^c, \dot{\Theta}_{\Pi a}) &\triangleq (1 - \hat{P}_{2r})(1 - \hat{P}_{r2}) \sum_{k=2t+1}^{n-1} \beta_k(p_{1r}) \sum_{m=\max(0, t+1-(n-k))}^t \sum_{i=t+1}^{m+(n-k)} \binom{k}{m} \binom{n-k}{i-m} \\ &\quad \times p_{\Pi}^{n-k-(i-m)} p_{12}^m \sum_{g=0}^{t-m} \sum_{j=0}^g \binom{k-m}{j} p_{\Pi}^j q_{\Pi}^{k-m-j} \binom{i-m}{g-j} \ddot{p}_{\Pi}^{g-j} \ddot{q}_{\Pi}^{i-m-(g-j)} \end{aligned} \quad (29)$$

where  $\beta_k(p_{1r})$  is given by (15), and

$$\begin{aligned} p_{\Pi} &= 1 - p_{12} \\ \dot{p}_{\Pi} &= Q(-\sqrt{2h^2E}) - Q\left(\sqrt{2E}\left(\frac{\mathcal{L}}{4hE} - h\right)\right), \quad \dot{q}_{\Pi} = Q\left(\sqrt{2E}\left(\frac{\mathcal{L}}{4hE} - h\right)\right) \\ \ddot{p}_{\Pi} &= Q\left(\sqrt{2E}\left(\frac{\mathcal{L}}{4hE} + h\right)\right), \quad \ddot{q}_{\Pi} = Q(\sqrt{2h^2E}) - Q\left(\sqrt{2E}\left(\frac{\mathcal{L}}{4hE} + h\right)\right). \end{aligned} \quad (30)$$

*Proof:* By using the inequality  $\hat{P}_{1r} \geq \tilde{P}_{1r} = \sum_{k=2t+1}^n \beta_k(p_{1r})$  given by Lemma 1, we have

$$P(\mathcal{E}^c, \dot{\Theta}_{\Pi a}) \geq (1 - \hat{P}_{2r})(1 - \hat{P}_{r2}) \sum_{k=2t+1}^{n-1} \beta_k(p_{1r}) P(\mathcal{E}^c, d(\mathbf{y}_{12}) \geq t+1 \mid d(\hat{\mathbf{x}}_{1r}) = k, \dot{\mathcal{E}}_{2r}^c, \dot{\mathcal{E}}_{r2}^c). \quad (31)$$

Note  $k = n$  term is omitted at (31) because of  $P(\mathcal{E}^c, \dot{\mathcal{E}}_{12} \mid d(\hat{\mathbf{x}}_{1r}) = n, \dot{\mathcal{E}}_{2r}^c, \dot{\mathcal{E}}_{r2}^c) = 0$ . In this subcase  $\dot{\Theta}_{\Pi a}$ , we have  $\mathcal{D}(\tilde{\mathbf{y}}_{r2}) = \hat{\mathbf{x}}_{1r}$  because  $\dot{\mathcal{E}}_{2r}^c \Rightarrow \mathcal{D}(\tilde{\mathbf{y}}_{r2}) = \hat{\mathbf{x}}_{1r} \circ \hat{\mathbf{x}}_{2r} \circ \mathbf{x}_2$  and  $\dot{\mathcal{E}}_{r2}^c \Rightarrow \hat{\mathbf{x}}_{2r} = \mathbf{x}_2 = \mathbf{1}$ . Then by using the fact and the inequality  $P(\mathcal{E}^c) \geq P(d(\ell_d) \leq t)$  with  $\ell_d = 4h\sqrt{E}\mathbf{y}_{12} + \hat{\mathbf{x}}_{1r}\mathcal{L}$ , we have

$$P(\mathcal{E}^c, d(\mathbf{y}_{12}) \geq t+1 \mid d(\hat{\mathbf{x}}_{1r}) = k, \dot{\mathcal{E}}_{2r}^c, \dot{\mathcal{E}}_{r2}^c) \geq P(d(\ell_d) \leq t, d(\mathbf{y}_{12}) \geq t+1 \mid d(\hat{\mathbf{x}}_{1r}) = k, \mathcal{D}(\tilde{\mathbf{y}}_{r2}) = \hat{\mathbf{x}}_{1r}). \quad (32)$$

Given  $d(\hat{\mathbf{x}}_{1r}) = k$  and  $d(\mathbf{y}_{12}) = i$ , as shown in Fig. 3, the inequality  $d(\ell_d) \leq t$  holds only if the cardinality of  $\{\tau : \hat{x}_{1r,\tau} = -1, y_{12,\tau} < 0\}$  is no larger than  $t$ , i.e.,  $m \leq t$ , and among the bits  $\{\tau : \hat{x}_{1r,\tau} = -1, y_{12,\tau} > 0\}$  and  $\{\tau : \hat{x}_{1r,\tau} = 1, y_{12,\tau} < 0\}$  there are only  $g \leq t-m$  erroneous channel bits. Suppose that  $j$  erroneous bits are within the set  $\{\tau : \hat{x}_{1r,\tau} = -1, y_{12,\tau} > 0\}$ , and the rest  $g-j$  erroneous bits are within the set  $\{\tau : \hat{x}_{1r,\tau} = 1, y_{12,\tau} < 0\}$ . For the  $\tau$ -th channel bit, let  $p_{\Pi} = P(y_{12,\tau} > 0) = 1 - p_{12}$ , and  $q_{\Pi} = P(y_{12,\tau} < 0) = p_{12}$ , and  $\dot{p}_{\Pi} = P(\ell_{d,\tau} < 0, y_{12,\tau} > 0 \mid \hat{x}_{1r,\tau} = -1) = \Pr(4h\sqrt{E}y_{12,\tau} - \mathcal{L} < 0, y_{12,\tau} > 0)$ , and  $\dot{q}_{\Pi} = P(\ell_{d,\tau} > 0, y_{12,\tau} > 0 \mid \hat{x}_{1r,\tau} = -1) = \Pr(4h\sqrt{E}y_{12,\tau} - \mathcal{L} > 0, y_{12,\tau} > 0)$ , and  $\ddot{p}_{\Pi} = P(\ell_{d,\tau} < 0, y_{12,\tau} < 0 \mid \hat{x}_{1r,\tau} = 1) = \Pr(4h\sqrt{E}y_{12,\tau} + \mathcal{L} < 0, y_{12,\tau} < 0)$  and  $\ddot{q}_{\Pi} = P(\ell_{d,\tau} > 0, y_{12,\tau} < 0 \mid \hat{x}_{1r,\tau} = 1) = \Pr(4h\sqrt{E}y_{12,\tau} + \mathcal{L} > 0, y_{12,\tau} < 0)$ . Then, by summing up the correct decoding probability  $p_{\Pi}^{n-k-(i-m)} q_{\Pi}^m p_{\Pi}^j q_{\Pi}^{k-m-j} \ddot{p}_{\Pi}^{g-j} \ddot{q}_{\Pi}^{i-m-(g-j)}$  over the combinations of the 5-tuples  $(k, i, m, g, j)$ ,

we have

$$\begin{aligned}
& P(d(\ell_d) \leq t, d(\mathbf{y}_{12}) \geq (t+1) \mid d(\hat{\mathbf{x}}_{1r}) = k, \mathfrak{D}(\tilde{\mathbf{y}}_{r2}) = \hat{\mathbf{x}}_{1r}) \\
&= \sum_{m=\max(0, t+1-(n-k))}^t \sum_{i=t+1}^{m+(n-k)} \binom{k}{m} \binom{n-k}{i-m} \\
&\quad \times p_{\Pi}^{n-k-(i-m)} p_{12}^m \sum_{g=0}^{t-m} \sum_{j=0}^g \binom{k-m}{j} \dot{p}_{\Pi}^j \dot{q}_{\Pi}^{k-m-j} \binom{i-m}{g-j} \ddot{p}_{\Pi}^{g-j} \ddot{q}_{\Pi}^{i-m-(g-j)}. \quad (33)
\end{aligned}$$

By substituting (33) into (32) and then into (31), we get (29), and thus prove the theorem.  $\blacksquare$

Based on the theorem and  $P(\dot{\Theta}_{\text{II}a}) = \widehat{P}_{12} \widehat{P}_{1r} (1 - \widehat{P}_{2r})(1 - \widehat{P}_{r2})$ , we find that  $P(\mathcal{E}, \dot{\Theta}_{\text{II}a})$  is upper bounded by

$$\widehat{P}(\mathcal{E}, \dot{\Theta}_{\text{II}a}) \triangleq \widehat{P}_{12} \widehat{P}_{1r} (1 - \widehat{P}_{2r})(1 - \widehat{P}_{r2}) - \check{P}(\mathcal{E}^c, \dot{\Theta}_{\text{II}a}). \quad (34)$$

2)  $P(\mathcal{E}, \dot{\Theta}_{\text{II}b})$ : We express  $P(\mathcal{E}, \dot{\Theta}_{\text{II}b}) = P(\dot{\Theta}_{\text{II}b}) - P(\mathcal{E}^c, \dot{\Theta}_{\text{II}b})$ , and lower bound  $P(\mathcal{E}^c, \dot{\Theta}_{\text{II}b})$  by the following corollary.

*Corollary 1.* Given a hard decoder of SPF  $W_k$ ,  $P(\mathcal{E}^c, \dot{\Theta}_{\text{II}b})$  is lower bounded by

$$\begin{aligned}
\check{P}(\mathcal{E}^c, \dot{\Theta}_{\text{II}b}) &\triangleq (1 - \widehat{P}_{1r})(1 - \widehat{P}_{r2}) \sum_{k=2t+1}^{n-1} \beta_k(p_{2r}) \sum_{m=\max(0, t+1-(n-k))}^t \sum_{i=t+1}^{m+(n-k)} \binom{k}{m} \binom{n-k}{i-m} \\
&\quad \times p_{\Pi}^{n-k-(i-m)} p_{12}^m \sum_{g=0}^{t-m} \sum_{j=0}^g \binom{k-m}{j} \dot{p}_{\Pi}^j \dot{q}_{\Pi}^{k-m-j} \binom{i-m}{g-j} \ddot{p}_{\Pi}^{g-j} \ddot{q}_{\Pi}^{i-m-(g-j)}. \quad (35)
\end{aligned}$$

*Proof:* This corollary is a direct use of Theorem 3 by exchanging the roles of  $\mathcal{E}_{1r}$  and  $\mathcal{E}_{2r}$ .  $\blacksquare$

Thus, by the corollary,  $P(\mathcal{E}, \dot{\Theta}_{\text{II}b})$  is upper bounded by

$$\widehat{P}(\mathcal{E}, \dot{\Theta}_{\text{II}b}) \triangleq \widehat{P}_{12} (1 - \widehat{P}_{1r}) \widehat{P}_{2r} (1 - \widehat{P}_{r2}) - \check{P}(\mathcal{E}^c, \dot{\Theta}_{\text{II}b}). \quad (36)$$

3) *Upper bound of  $P_{\text{TW-lbSF}}^{(\mathbf{x}_1)}$* : Finally, recall

$$P(\dot{\Theta}_{\text{II}c}) = \widehat{P}_{12} \widehat{P}_{1r} \widehat{P}_{2r} (1 - \widehat{P}_{r2}) \quad (37)$$

and

$$P(\dot{\Theta}_{\text{III}}) = \widehat{P}_{12} (1 - (1 - \widehat{P}_{1r})(1 - \widehat{P}_{2r})) \widehat{P}_{r2}. \quad (38)$$

By substituting (19) into (24), and (29) into (34), and (35) into (36), we get  $\widehat{P}(\mathcal{E}, \dot{\Theta}_{\text{I}})$ ,  $\widehat{P}(\mathcal{E}, \dot{\Theta}_{\text{II}a})$  and  $\widehat{P}(\mathcal{E}, \dot{\Theta}_{\text{II}b})$ , respectively. Then, by substituting these upper bounds, along with (37) and (38), into (28), we obtain an upper bound for  $P(\mathcal{E}, \Theta_{\text{I}}, \Theta_{\text{II}}, \Theta_{\text{III}})$ , i.e.,  $P_{\text{TW-lbSF}}^{(\mathbf{x}_1)}$ , by

$$\widehat{P}_{\text{TW-lbSF}}^{(\mathbf{x}_1)} \triangleq \widehat{P}(\mathcal{E}, \dot{\Theta}_{\text{I}}) + \widehat{P}(\mathcal{E}, \dot{\Theta}_{\text{II}a}) + \widehat{P}(\mathcal{E}, \dot{\Theta}_{\text{II}b}) + P(\dot{\Theta}_{\text{II}c}) + P(\dot{\Theta}_{\text{III}}). \quad (39)$$

*Remark 1 (Performance of Perfect Codes).* For perfect codes, the sphere decoder is exactly the optimal hard decoder, and hence  $W_{2t+1} = A_{2t+1} \binom{2t+1}{t} = \binom{n}{t+1}$  and  $W_k = 0$  for all  $k > 2t + 1$ , the lower bound given by (14) and the upper bound given by (13) are equal, and also equal to the dedicated lower bound given by (15), so,  $P(\mathcal{E}, \Theta_I, \Theta_{II}, \Theta_{III}) = P(\mathcal{E}, \dot{\Theta}_I, \dot{\Theta}_{II}, \dot{\Theta}_{III})$ . Furthermore, the upper bound derived for the 1bSF protocol is asymptotically tight because the results for  $P(\mathcal{E}, \dot{\Theta}_I)$ ,  $P(\mathcal{E}, \dot{\Theta}_{IIa})$  and  $P(\mathcal{E}, \dot{\Theta}_{IIb})$  are exact, while  $P(\dot{\Theta}_{IIIc})$  and  $P(\dot{\Theta}_{III})$  are asymptotically negligible.

#### E. Asymptotic performance analysis of TW-1bSF and TW-SDF protocols

The analytical comparison of the BLER performance of the TW-1bSF and the TW-SDF protocols is prohibitive because of their sophisticated expressions. To gain insights, we derive and compare their asymptotic performance with respect to  $E \rightarrow \infty$  and  $E_r \rightarrow \infty$ . In the following analysis,  $f_A(E, E_r)$  and  $f_B(E, E_r)$  are said to be asymptotically equal with respect to  $E$  and  $E_r$ , denoted as  $f_A(E, E_r) \sim f_B(E, E_r)$ , if  $\log_{E \rightarrow \infty, E_r \rightarrow \infty} f_A(E, E_r) / f_B(E, E_r) = 1$ . We first present a lemma which will be used in the asymptotic analysis afterwards.

*Lemma 3.* Let  $z_1 \sim \mathcal{N}(\mu_1 \sqrt{E}, 1)$  and  $z_2 \sim \mathcal{N}(\mu_2 \sqrt{E}, 1)$  be two independent Gaussian random variables. If  $\Lambda_1 = \Pr(z_1 + z_2 > 0, z_1 > 0, z_2 < 0)$  and  $\Lambda_2 = \Pr(z_1 + z_2 < 0, z_1 > 0, z_2 < 0)$ , then  $\Lambda_1 / \Lambda_2 \rightarrow \infty$  as  $E \rightarrow \infty$ .

*Proof:* Let  $0 < \epsilon < \min\{\mu_1, \mu_2\}$  be a small positive value. We can lower bound  $\Lambda_1$  by

$$\begin{aligned} \Lambda_1 &= \left( \int_{-(\mu_1 - \epsilon)\sqrt{E}}^0 + \int_{-\infty}^{-(\mu_1 - \epsilon)\sqrt{E}} \right) \left( 1 - Q((\mu_1 \sqrt{E} + z_2)\sqrt{2}) \right) f(z_2) dz_2 \\ &\geq \left( 1 - Q(\epsilon\sqrt{2E}) \right) \left( Q(\mu_2 \sqrt{2E}) - Q((\mu_2 + \mu_1 - \epsilon)\sqrt{2E}) \right), \end{aligned}$$

which is derived by lower bounding the second integral with 0, and the first integral with the fact that

$$\left( 1 - Q((\mu_1 \sqrt{E} + z_2)\sqrt{2}) \right) \big|_{z_2 \in [0, -(\mu_1 - \epsilon)\sqrt{E}]} \geq \left( 1 - Q((\mu_1 \sqrt{E} + z_2)\sqrt{2}) \right) \big|_{z_2 = -(\mu_1 - \epsilon)\sqrt{E}}.$$

On the other hand, we can upper bound  $\Lambda_2$  by

$$\begin{aligned} \Lambda_2 &= \left( \int_{-(\mu_1 - \epsilon)\sqrt{E}}^0 + \int_{-\infty}^{-(\mu_1 - \epsilon)\sqrt{E}} \right) \left( Q((\mu_1 \sqrt{E} + z_2)\sqrt{2}) - Q(\mu_1 \sqrt{2E}) \right) f(z_2) dz_2 \\ &\leq \left( Q(\epsilon\sqrt{2E}) - Q(\mu_1 \sqrt{2E}) \right) \left( Q(\mu_2 \sqrt{2E}) - Q((\mu_2 + \mu_1 - \epsilon)\sqrt{2E}) \right) + Q(-\mu_1 \sqrt{2E}) Q((\mu_2 + \mu_1 - \epsilon)\sqrt{2E}). \end{aligned}$$

So when  $E \rightarrow \infty$ , we have  $Q(-\mu_1 \sqrt{2E}) \rightarrow 1$  and  $Q(\epsilon\sqrt{2E}) / Q(\mu_1 \sqrt{2E}) \rightarrow \infty$  and  $Q(\mu_2 \sqrt{2E}) / Q((\mu_2 + \mu_1 - \epsilon)\sqrt{2E}) \rightarrow \infty$ , and thus

$$\frac{\Lambda_1}{\Lambda_2} \geq \frac{(1 - Q(\epsilon\sqrt{2E})) Q(\mu_2 \sqrt{2E})}{Q(\epsilon\sqrt{2E}) Q(\mu_2 \sqrt{2E}) + Q((\mu_2 + \mu_1 - \epsilon)\sqrt{2E})} \rightarrow \infty,$$



which proves the lemma.  $\blacksquare$

1) *Asymptotic performance of TW-SDF*: With respect to  $E \rightarrow \infty$  and  $E_r \rightarrow \infty$ , i.e.,  $p_i \rightarrow 0$ , the bounds (13) and (14) have the asymptotical forms  $\hat{P}_i \sim \binom{n}{t+1} p_i^{t+1}$  and  $\check{P}_i' \sim A_{2t+1} \binom{2t+1}{t} p_i^{t+1}$ . We start with the analysis of  $\hat{P}(\mathcal{E}, \Theta_I)$  in (27) for TW-SDF, together with  $\hat{P}(\mathcal{E}, \Theta_I)$  in (24) for TW-1bSF.

Case I): Since  $\hat{P}_{12} (1 - \hat{P}_{1r})(1 - \hat{P}_{2r}) \hat{P}_{r2} \sim \hat{P}_{12} (1 - \check{P}_{1r}')(1 - \check{P}_{2r}') \hat{P}_{r2} \sim \hat{P}_{12} \hat{P}_{r2} \sim \binom{n}{t+1} p_{12}^{t+1} \binom{n}{t+1} p_{r2}^{t+1}$ , it gives that  $\hat{P}(\mathcal{E}, \Theta_I) \sim \hat{P}(\mathcal{E}, \Theta_I) \sim \binom{n}{t+1} p_{12}^{t+1} \binom{n}{t+1} p_{r2}^{t+1} - \check{P}(\mathcal{E}^c, \Theta_I)$ . In  $\check{P}(\mathcal{E}^c, \Theta_I)$  given by (19), it is clear from Lemma 3 that  $\dot{q}_I/\dot{p}_I \rightarrow \infty$  and  $\ddot{q}_I/\ddot{p}_I \rightarrow \infty$ , and hence  $\dot{q}_I \sim p_{r2}(1 - p_{12}) \sim p_{r2}$  and  $\ddot{q}_I \sim p_{12}(1 - p_{r2}) \sim p_{12}$ . So  $\check{P}(\mathcal{E}^c, \Theta_I)$  is dominated by the terms in which  $\dot{p}_I^j = \ddot{p}_I^{g-j} = 1$ , i.e.,  $j = g = 0$ , and  $i = t + 1$  and  $k = t + 1$ . Therefore

$$\begin{aligned} \check{P}(\mathcal{E}^c, \Theta_I) &\sim \binom{n}{t+1} \sum_{m=0}^t \binom{t+1}{m} \binom{n-(t+1)}{t+1-m} p_{r2}^m p_{12}^m \dot{q}_I^{t+1-m} \ddot{q}_I^{t+1-m} \\ &\sim \binom{n}{t+1} p_{r2}^{t+1} p_{12}^{t+1} \sum_{m=0}^t \binom{t+1}{m} \binom{n-(t+1)}{t+1-m}. \end{aligned} \quad (40)$$

And so

$$\begin{aligned} \hat{P}(\mathcal{E}, \Theta_I) \sim \hat{P}(\mathcal{E}, \Theta_I) &\sim \binom{n}{t+1}^2 p_{12}^{t+1} p_{r2}^{t+1} - \binom{n}{t+1} p_{r2}^{t+1} p_{12}^{t+1} \sum_{m=0}^t \binom{t+1}{m} \binom{n-(t+1)}{t+1-m} \\ &\sim p_{12}^{t+1} p_{r2}^{t+1} \binom{n}{t+1}. \end{aligned} \quad (41)$$

To derive the expression above, we use the fact  $\binom{n}{t+1} - \sum_{r=0}^t \binom{t+1}{r} \binom{n-(t+1)}{t+1-r} = \binom{t+1}{t+1} = 1$ .

Then, by substituting this asymptotical form and  $P(\Theta_{II} \cup \Theta_{III}) \sim \hat{P}_{12} (\hat{P}_{1r} + \hat{P}_{2r})$  into (25), we obtain

$$\begin{aligned} \hat{P}_{\text{TW-SDF}}^{(\mathbf{x}_1)} &\sim \hat{P}_{\text{TW-SDF,asym}}^{(\mathbf{x}_1)} \\ &\triangleq \binom{n}{t+1} p_{12}^{t+1} p_{r2}^{t+1} + \binom{n}{t+1}^2 [p_{1r}^{t+1} + p_{2r}^{t+1}] p_{12}^{t+1}. \end{aligned} \quad (42)$$

2) *Asymptotic performance of TW-1bSF*: Among the terms of  $\hat{P}_{\text{TW-1bSF}}^{(\mathbf{x}_1)}$  in (39), we have analyzed  $\hat{P}(\mathcal{E}, \Theta_I)$  in (41). In the following, we derive the rest terms for Case II and Case III.

Case II): In  $\hat{P}(\mathcal{E}, \Theta_{IIa})$  given by (29), it is worthwhile to note that  $\dot{q}_{II}$  is the probability that an erroneous bit  $i$  from the relay with  $\hat{x}_{1r,i} = -1$  can be corrected by  $y_{12,i} > 0$ , as shown in Fig. 3. To optimize the performance of TW-1bSF,  $\dot{q}_{II}$  should be maximized. Therefore it is desirable  $\dot{q}_{II} > \dot{p}_{II}$ , and thus the expected condition  $\frac{\mathcal{L}}{4hE} \sim \frac{\min(h_1^2, h_2^2)}{4h} < h$ . Given the condition, now, it gives  $\dot{q}_{II}/\dot{p}_{II} \rightarrow \infty$  as  $E \rightarrow \infty$ , so  $\dot{q}_{II} \sim 1$ . Note  $\frac{\mathcal{L}}{4hE} \sim \frac{\min(h_1^2, h_2^2)}{4h} < h$  could be satisfied when  $h = h_1 = h_2 = 1$  for instance. On the other hand, it is clear  $\ddot{q}_{II}/\ddot{p}_{II} \rightarrow \infty$ , and so  $\ddot{q}_{II} \sim Q(\sqrt{2h^2E}) = p_{12}$ . By using these asymptotic results for  $\dot{q}_{II}$  and  $\ddot{q}_{II}$ , we conclude that in  $\hat{P}(\mathcal{E}, \Theta_{IIa})$  the dominant events happen when  $i = t + 1$ ,  $g = 0$ ,

$j = 0$ , for all  $k$  with  $W_k \geq 0$ , so we have  $P(d(\ell_d) \leq t, d(\mathbf{y}_{12}) \geq t+1 \mid d(\hat{\mathbf{x}}_{1r}) = k, \mathfrak{D}(\tilde{\mathbf{y}}_{r2}) = \hat{\mathbf{x}}_{1r}) \sim \sum_{m=0}^t \binom{k}{m} \binom{n-k}{t+1-m} p_{12}^{t+1} \dot{q}_{\text{II}}^{k-m}$  at (33). Therefore, by substituting it and  $\beta_k(p_{1r}) \sim W_k p_{1r}^{t+1}$  into (29), and then into (34), we get

$$\begin{aligned} \hat{P}(\mathcal{E}, \dot{\Theta}_{\text{IIa}}) &\sim \binom{n}{t+1}^2 p_{12}^{t+1} p_{1r}^{t+1} - \sum_{k=2t+1}^n W_k p_{1r}^{t+1} p_{12}^{t+1} \sum_{m=0}^t \binom{k}{m} \binom{n-k}{t+1-m} \dot{q}_{\text{II}}^{k-m} \\ &\sim p_{1r}^{t+1} p_{12}^{t+1} \left[ \binom{n}{t+1}^2 - \sum_{k=2t+1}^n W_k \left( \binom{n}{t+1} - \binom{k}{t+1} \right) \right]. \end{aligned} \quad (43)$$

The derivation of the last step uses the fact that  $\sum_{r=0}^t \binom{k}{r} \binom{n-k}{t+1-r} = \binom{n}{t+1} - \binom{k}{t+1}$ . By replacing  $p_{1r}$  with  $p_{2r}$  at the asymptotic equation above, we have

$$\hat{P}(\mathcal{E}, \dot{\Theta}_{\text{IIb}}) \sim p_{2r}^{t+1} p_{12}^{t+1} \left[ \binom{n}{t+1}^2 - \sum_{k=2t+1}^n W_k \left( \binom{n}{t+1} - \binom{k}{t+1} \right) \right]. \quad (44)$$

Given at (37),  $P(\dot{\Theta}_{\text{IIc}})$  has the asymptotic form  $P(\dot{\Theta}_{\text{IIc}}) \sim \binom{n}{t+1}^2 (p_{1r} p_{2r})^{t+1} p_{12}^{t+1}$ . By comparing  $P(\dot{\Theta}_{\text{IIc}}) \sim O((p_{1r} p_{2r})^{t+1} p_{12}^{t+1})$  with  $\hat{P}(\mathcal{E}, \dot{\Theta}_{\text{IIa}})$  and  $\hat{P}(\mathcal{E}, \dot{\Theta}_{\text{IIb}})$ , which are  $O(p_{12}^{t+1} p_{1r}^{t+1})$  and  $O(p_{12}^{t+1} p_{2r}^{t+1})$ , it is clear that  $P(\dot{\Theta}_{\text{IIc}})$  is asymptotically negligible.

Case III): By using the similar arguments for  $P(\dot{\Theta}_{\text{IIc}})$ , we also find that  $P(\dot{\Theta}_{\text{III}})$  is asymptotically negligible, compared with  $\hat{P}(\mathcal{E}, \dot{\Theta}_{\text{I}})$ ,  $\hat{P}(\mathcal{E}, \dot{\Theta}_{\text{IIa}})$  and  $\hat{P}(\mathcal{E}, \dot{\Theta}_{\text{IIb}})$ .

By substituting (41), (43) and (44) into (39), we obtain the asymptotic form of  $\hat{P}_{\text{TW-1bSF}}^{(\mathbf{x}_1)}$

$$\hat{P}_{\text{TW-1bSF,asym}}^{(\mathbf{x}_1)} \triangleq p_{12}^{t+1} p_{r2}^{t+1} \binom{n}{t+1} + (p_{1r}^{t+1} + p_{2r}^{t+1}) p_{12}^{t+1} \left[ \binom{n}{t+1}^2 - \sum_{k=2t+1}^n W_k \left( \binom{n}{t+1} - \binom{k}{t+1} \right) \right]. \quad (45)$$

3) *Asymptotic Performance Comparison:* Comparing the asymptotic performance of SDF and 1bSF given by (42) and (45), respectively, we note that they have the same first term, but the second term of (45) is smaller than the second term of (42). This observation means that TW-1bSF has performance gain over TW-SDF if the error events of Case II is dominant over the error events of Case I, which is satisfied if  $p_{r2} \leq p_{1r}$  or  $p_{r2} \leq p_{2r}$  for instance. Now, suppose  $E_r = E$  and  $h = h_1 = h_2 = 1$ , which satisfy both conditions  $p_{r2} \leq p_{2r}$  and  $\dot{q}_{\text{II}} \sim 1$ , then we have the asymptotic performance gain of the TW-1bSF protocol over the TW-SDF protocol

$$\frac{\hat{P}_{\text{TW-SDF,asym}}^{(\mathbf{x}_1)}}{\hat{P}_{\text{TW-1bSF,asym}}^{(\mathbf{x}_1)}} = \binom{n}{t+1}^2 / \left[ \binom{n}{t+1}^2 - \sum_{k=2t+1}^n W_k \left( \binom{n}{t+1} - \binom{k}{t+1} \right) \right]. \quad (46)$$

If perfect codes like Hamming codes are applied, the expression of the asymptotic gain can be further simplified by noting that  $W_{2t+1} = \binom{n}{t+1}$  and  $W_k = 0$  for  $k = 0$  and all  $k > 2t+1$ . The simplified

asymptotic gain by perfect codes is expressed by

$$\frac{\widehat{P}_{\text{TW-SDF,pc,asym}}^{(\mathbf{x}_1)}}{\widehat{P}_{\text{TW-1bSF,pc,asym}}^{(\mathbf{x}_1)}} = \binom{n}{t+1} / \binom{2t+1}{t+1}. \quad (47)$$

This asymptotic gain shows that the asymptotic performance gain increases as the codeword length.

*Remark 2* (Redesign of Reliability Value). According to the asymptotic analysis above, in order to maximize the performance gain of the TW-1bSF protocol over the TW-SDF protocol, two conditions are desired, which are: (1)  $\mathcal{L}/4hE < h$  such that  $\dot{q}_{\text{II}} \sim 1$ ; (2)  $p_{r2} \leq p_{1r}$  or  $p_{r2} \leq p_{2r}$  such that the rate of error events at Case II dominates over the rate of error events at Case I. Condition (2) can be easily satisfied since  $p_{r2} = p_{2r}$  when  $E_r = E$ . As for Condition (1), the original design,  $\mathcal{L} \sim \min(h_1^2 E, h_2^2 E)$ , has no guarantee on the validness of the condition. For instance, the condition is violated if  $h = 1$ ,  $h_1 = h_2 = 2$ . In order to satisfy the condition, the reliability can be redesigned as

$$\mathcal{L}^* = \min(\log \frac{1-p_{1r}}{p_{1r}}, \log \frac{1-p_{2r}}{p_{2r}}, \log \frac{1-p_{12}}{p_{12}}). \quad (48)$$

The superiority of this design will be illustratively verified in the following section. On the other hand, the design of  $\mathcal{L}^-$  is not as critical as  $\mathcal{L}$  since the asymptotic performance is irrelevant to  $\mathcal{L}^-$ .

*Remark 3* (Decoding Energy Consumption Comparison). Let  $\epsilon$  be the energy consumption of decoding a codeword at  $S2$ . Given  $\mathbf{y}_{12}$  is in error, in the TW-SDF protocol, if  $R$  correctly decodes the received signals and forwards the network-coded word,  $S2$  will consume  $3\epsilon$  for decoding  $\mathbf{y}_{12}$ ,  $\tilde{\mathbf{y}}_{r2}$  and  $(4h\sqrt{E}\mathbf{y}_{12} + 4h_2\sqrt{E_r}\tilde{\mathbf{y}}_{r2})$ ; otherwise,  $R$  forwards nothing, and  $S2$  merely decodes  $\mathbf{y}_{12}$  consuming  $\epsilon$ . Therefore, the decoding energy consumed at  $S2$  in the TW-SDF protocol is

$$E_{\text{TW-SDF}} = \epsilon P(\tilde{e}_s) + 3\epsilon P(\mathcal{E}_{12}, \mathcal{E}_{1r}^c, \mathcal{E}_{2r}^c) + \epsilon (P(\mathcal{E}_{12}) - P(\mathcal{E}_{12}, \mathcal{E}_{1r}^c, \mathcal{E}_{2r}^c)) \approx \epsilon(1 - \widehat{P}_{12}) + 3\epsilon P(\Theta_{\text{I}}) + \epsilon P(\Theta_{\text{II}} \cup \Theta_{\text{III}}). \quad (49)$$

In the TW-1bSF protocol, if  $\mathbf{y}_{12}$  is in error, no matter whether  $R$  could correctly decode the received signals, it would forward  $\mathbf{x}_r$ , so  $S2$  consumes  $3\epsilon P(\Theta_{\text{II}} \cup \Theta_{\text{III}})$ . The decoding energy consumed in the 1bSF protocol is given by

$$E_{\text{TW-1bSF}} \approx \epsilon(1 - \widehat{P}_{12}) + 3\epsilon P(\Theta_{\text{I}}) + 3\epsilon P(\Theta_{\text{II}} \cup \Theta_{\text{III}}). \quad (50)$$

Since  $P(\Theta_{\text{I}}) \sim \widehat{P}_{12}\widehat{P}_{r2}$  and  $P(\mathcal{E}, \Theta_{\text{II}} \cup \Theta_{\text{III}}) \sim \widehat{P}_{12}(\widehat{P}_{1r} + \widehat{P}_{2r})$ , we have  $E_{\text{SDF}} \sim E_{\text{1bSF}} \sim \epsilon(1 - \widehat{P}_{12})$ , which means the 1bSF protocol consumes no more decoding energy than the SDF protocol if the SNR  $E/N_0$  is not too small.

## V. RESULTS AND DISCUSSIONS

The proposed TW-1bSF protocol is simple, in terms of signal processing complexity, and hence consumes relatively less power, so it is suitable to be incorporated into communication systems with a stringent power constraint such as Bluetooth (IEEE 802.15.1) [20], IEEE 802.15.6 [21], and WBAN (IEEE 802.15.6) [21] etc. The channel codes utilized by these simple communication standards therefore are the focus of this paper. As an example of perfect codes, (15,11) Hamming code is adopted at Bluetooth [20], and so is considered here. Let  $r_c$  be the code rate,  $E_b/N_0$  be the SNR of information bits, then the SNR of code bits is  $E/N_0 = r_c E_b/N_0$ . For sensor networks, all nodes can generally switch the mode between transmitter and relay, so it is normal to assume that  $E = E_r$ . In the rest of analysis, it is also assumed that  $h = h_1 = h_2 = 1$  unless otherwise specified. In this setting, both conditions preferred by TW-1bSF in Remark 2 are satisfied.

Given (15,11) Hamming code, Fig. 4 shows the simulated and analytical BLER performance for TW-SDF and TW-1bSF. It is observed that the simulation results are tightly bounded by the upper bound derived for TW-1bSF, and perfectly match with the derived upper bound for TW-SDF. This comparison verifies the performance analysis presented at Section IV. It also shows that given BLER ranging from  $10^{-2}$  to  $10^{-3}$ , the SNR gain of 1bSF over SDF is around 0.6 dB, which offers power saving by about 15%.

As an extension of Hamming code, BCH code is capable of correcting multiple error bits and hence also widely adopted for small-area communication systems [17], [21]. In the simulation work of this paper, BCH codeword is generated by a systematic encoder, and a received word is decoded by a Berlekamp-Massey decoder; both the encoder and the decoder are provided by Matlab Communication toolbox. It is worthwhile to note that in the Matlab BCH decoder, if a decoding failure is detected, then the first received  $k$  bits are decoded as the estimation of the transmitted information message if the code rate is  $k/n$ . Given a double-error-correcting (127,113) BCH code as an example, the SPF  $W_k$  associated with the Matlab encoder/decoder is shown in Table. I. For this code, the BLER results achieved by simulation and the analytical upper bound for TW-1bSF and TW-SDF are shown in Fig. 5. It is observed that the simulation results are tightly upper bounded by the derived bound for TW-SDF, and well bounded by the one for TW-1bSF, which is asymptotically tight. The simulation results also show that, given the BER range from  $10^{-2}$  to  $10^{-3}$ , the SNR gain offered by TW-1bSF is about 0.8 dB, which indicates the power saving by 20%. The figures demonstrate that the asymptotic tightness of the derived upper bounds for the TW-SDF and the TW-1bSF protocols is valid for any hard decoders with negligible  $W_0$ . This is because

these decoders can be approximated by the BDD which has  $W_0 = 0$ , and in which the upper bounds are achieved. Table. I shows that the applied Matlab BCH decoder has very small  $W_0$ .

Fig.6 displays the upper bound curves vs SNR to illustrate the impact of codeword length and error-correcting capability on the performance gain of TW-1bSF over TW-SDF. The performance comparison between (7,4) Hamming code and (127,113) Hamming code shows that given a fixed error-correcting capability  $t$ , the performance gain increases with the codeword length  $n$ . The comparison between (127,120) Hamming code and (127,113) BCH code shows that, given a fixed codeword length  $n$ , higher error-correcting capability  $t$  may however reduce the performance gain. Given BCH code, Fig. 7 quantitatively shows the asymptotic performance gain as a function of codeword length  $n$ , and error-correcting capability  $t$ , along with the simulated performance gain obtained when  $E_b/N_0 = 5$  (dB). In this figure, the asymptotical performance gain and the simulated performance gain demonstrate the similar functional trend over the error-correcting capability  $t$ , and both increase with the codeword length  $n$ , but not monotonically related with  $t$ .

Finally, Fig. 8 is presented to verify the conclusion in Remark 2 that the reliability design of (48) is better than (8). Fig. 8.a shows the BLER performance of these two reliability designs versus the channel gain of the direct link  $h$ , given fixed  $E_b/N_0 = 7$ dB. When  $h$  is small, which can be interpreted as there is no direct link, TW-SDF and TW-1bSF performs the same for both reliability designs. The reliability design (48) starts to show performance gain over TW-SDF when  $h = 0.4$ , and the performance gain keep increasing as  $h$  goes large. By contrast, the reliability design (8) does not start to show any performance gain over SDF until  $h = 0.5$ , in which Condition (2) starts to be satisfied. The performance improvement of the design (48) over the design (8) are appealing between  $h = 0.6$  and  $h = 0.7$ . Furthermore, Fig. 8.b shows the BLER performance of the two reliability designs vs. SNR, given  $h = 0.6$ . It shows that, among the BLER range from  $10^{-2}$  to  $10^{-4}$ , the TW-1bSF protocol based on the design (48) offers 0.6 dB gain over TW-SDF, which is much better than the negligible gain of the one based on the design (8).

## VI. CONCLUSIONS

In this paper, we propose the energy-efficient two-way one-bit soft forwarding (TW-1bSF) protocol to improve the TW-SDF protocol in DNC-based two-way relay networks, by forwarding the network-coded packet which contains erroneous bits, and would be discarded by the relay in the TW-SDF protocol. The key ingredient of the TW-1bSF protocol is to weight the erroneous network-coded packet with a reliability value. By careful designing the reliability parameter, the TW-1bSF protocol gains the attractive performance of the soft relaying approach while preserving the simplicity of the TW-SDF protocol.

We also derive tight upper bounds on the BLER of the TW-1bSF and the TW-SDF protocols when applying block codes and hard decoding, and verify these bounds by simulation. Our further analysis shows that the asymptotic performance gain of the TW-1bSF protocol over the TW-SDF protocol grows with the codeword length. This suggests that the TW-1bSF protocol can give an impressive performance improvement relative to the TW-SDF, especially, when a longer code is used.

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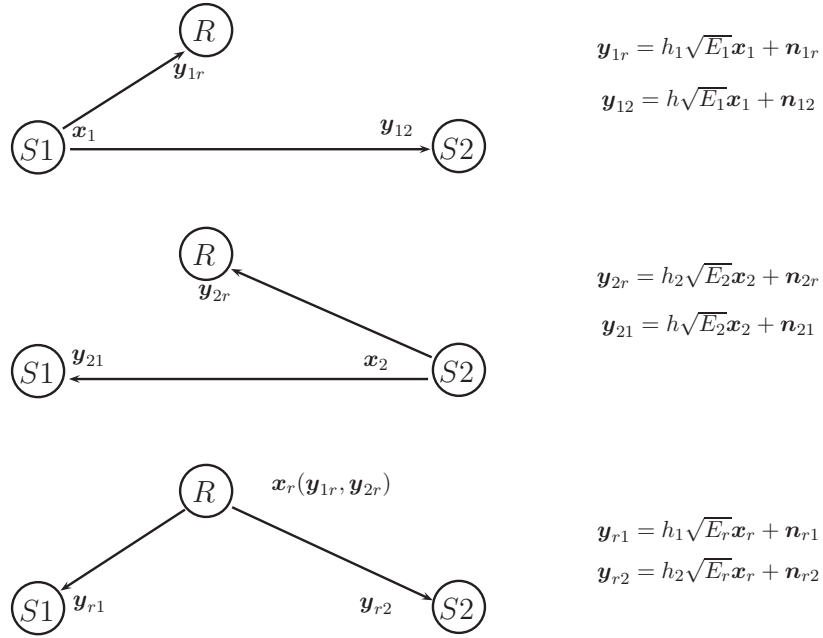


Figure 1. Two-way relaying based on direct network coding (DNC).



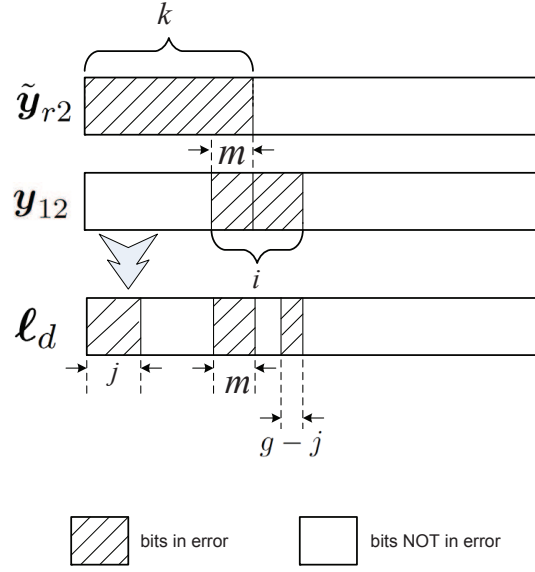


Figure 2. The alignment between  $\tilde{y}_{r2}$  and  $y_{12}$  for decoding  $\ell_d = 4h\sqrt{E}\mathbf{y}_{12} + 4h_2\sqrt{E_r}\tilde{\mathbf{y}}_{r2}$  in Case I. The erroneous channel bits are represented by shadow box, while the correct channel bits are represented by blank box.

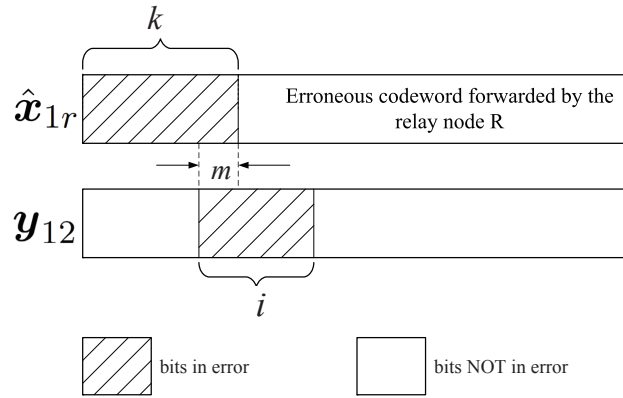


Figure 3. The error position alignment between the received signals from the  $R$ - $S2$  link and the  $S1$ - $S2$  link.

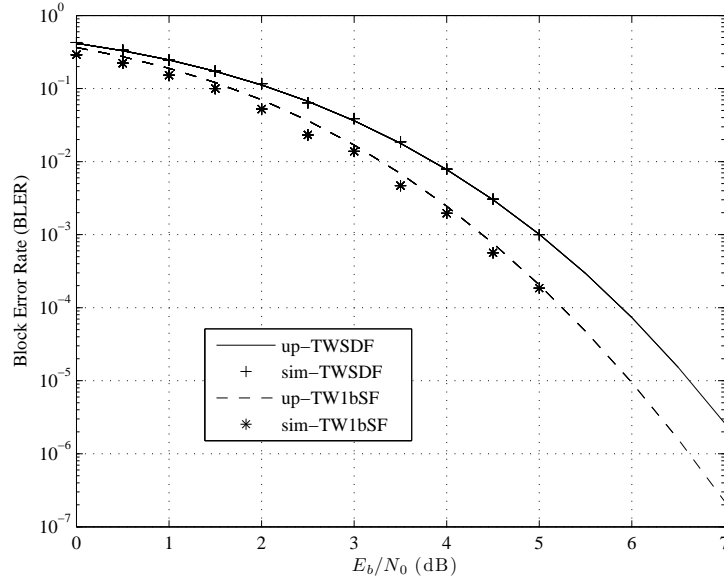


Figure 4. Simulation results vs analytical results of TW-1bSF and TW-SDF for the two-way network using (15,11) Hamming code. “sim” stands for simulation, while “up” stands for the derived BLER upper bound.

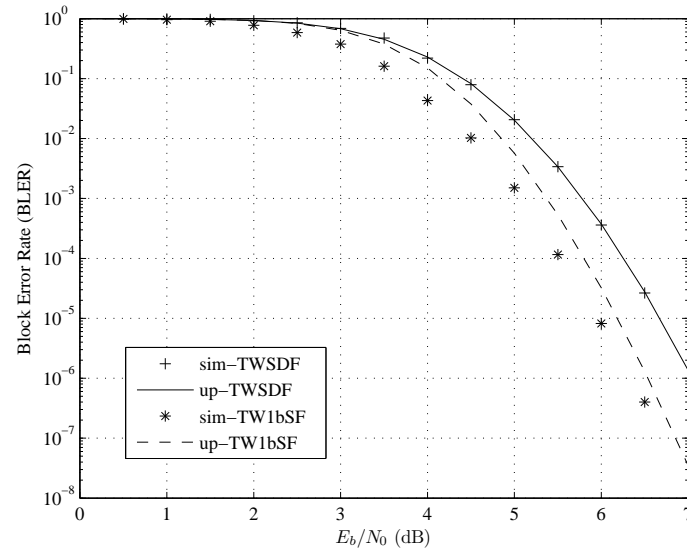


Figure 5. Performance of TW-SDF and TW-1bSF for the two-way network using (127,113) BCH code.

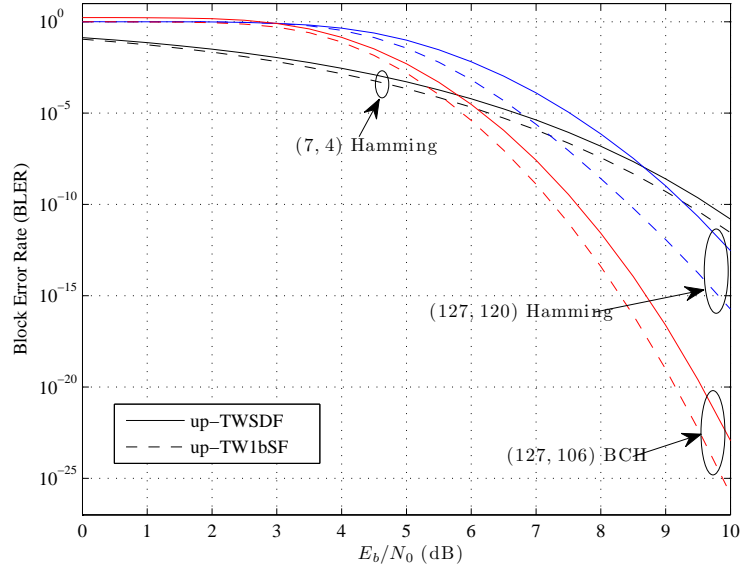


Figure 6. Performance comparison of TW-SDF and TW-1bSF for different codes in terms of block error rate (BLER).

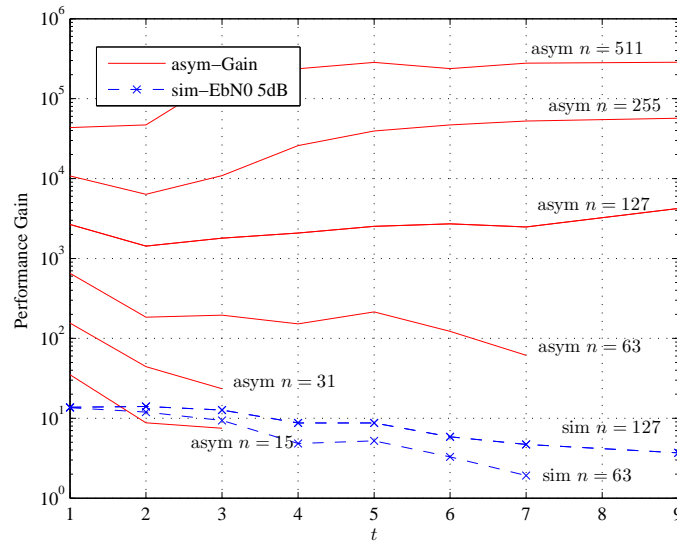
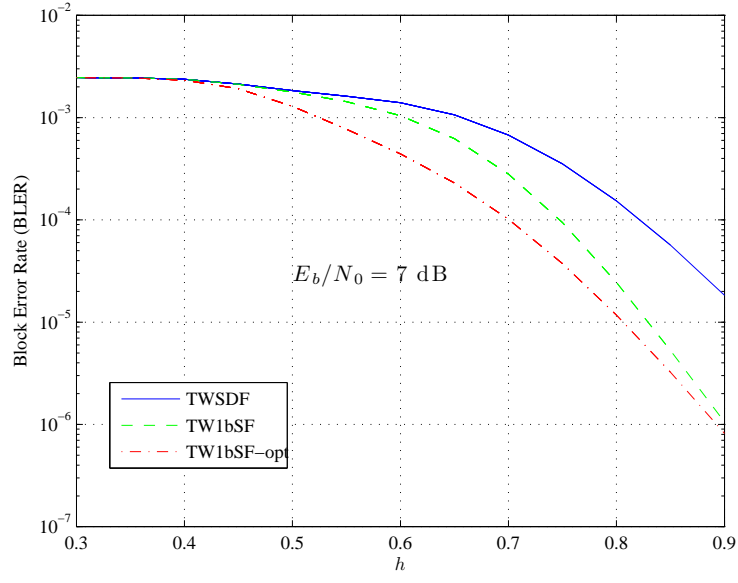
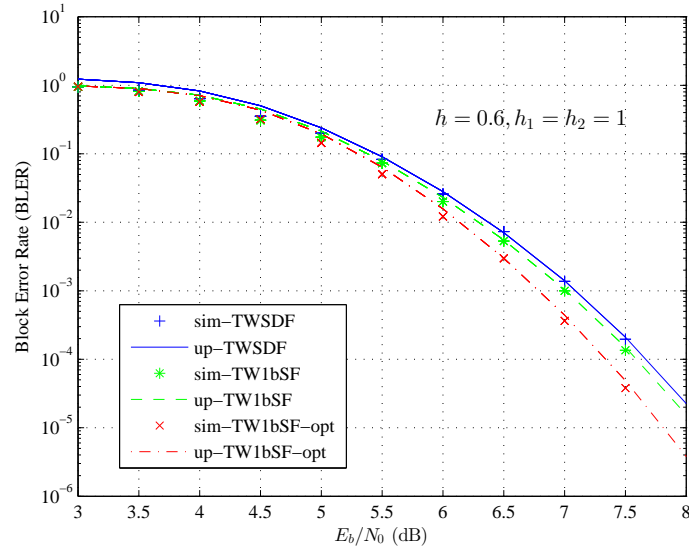


Figure 7. Asymptotic and simulated performance gain versus error-correcting capability  $t$  for a group of codeword length  $n$  when BCH code is applied.



(a)



(b)

Figure 8. (a), Comparison of reliability designs. (127,113) BCH code is employed. “1bSF-rel” stands for the result derived by using the reliability value design (48) while “1bSF” stands for the result derived by using the reliability value design (8); (b), BLER performance versus SNR given  $h = 0.6$  and  $h_1 = h_2 = 1$ .

$W_k$	$n = 15$	$n = 31$	$n = 63$	$n = 127$	$n = 255$	$n = 511$
$k = 0$	36	71	170	164	336	362
$k = d_{\min}$	281	2123	19316	161664	1351834	10949163
$k = d_{\min} + 1$	89	540	2557	5399	19172	42719
$k = d_{\min} + 2$	26	690	3600	15538	49781	147656
$k = d_{\min} + 3$	11	616	4102	25178	103059	416474
$k = d_{\min} + 4$	9	280	3916	32427	183177	833629
$k = d_{\min} + 5$	3	126	3550	34044	249573	1406293
$k = d_{\min} + 6$		41	1594	28283	266769	1887989
$k = d_{\min} + 7$		7	735	17866	222982	2050926
$k = d_{\min} + 8$		1	136	8791	156460	1828617
$k = d_{\min} + 9$			31	3277	83028	1301727
$k = d_{\min} + 10$			4	656	31614	746208
$k = d_{\min} + 11$				88	10672	340297
$k = d_{\min} + 12$					2387	116561
$k = d_{\min} + 13$					291	32983
$k = d_{\min} + 14$						6198
$k = d_{\min} + 15$						539
$k = d_{\min} + 16$						74

Table I

THE SPHERE PARTITION FUNCTION  $W_k$  FOR TWO-ERROR-CORRECTING ( $t = 2, d_{\min} = 5$ ) BCH CODES WITH CODE LENGTH  $n = 15, 31, 63, 127, 255, 511$ .